MAC 2312 — Calculus II

Guided Notes

Section 8.1

Arc Length

Definition

Let the function f be smooth on a closed interval $[a,b]$. The arc length of the graph of f from $A(a, f(a))$ to $B(b, f(b))$ is given by

$$L = \int_a^b \sqrt{1+(f'(x))^2}\, dx$$

use this version if the definition of the function is in the form $\boxed{y = f(x)}$

Alternately

Let the function g be defined by $x = g(y)$ where g is smooth on the closed interval $[c,d]$. The arc length of the graph of g from point $(g(c), c)$ to $(g(d), d)$ is given by

$$L = \int_c^d \sqrt{1+(g'(y))^2}\, dy$$

Format

$x = g(y)$

Example 1

If $f(x) = 3x^{2/3} - 10$, find the arc length of the graph of f from $(8,2)$ to $(27,17)$.

function format $y=f(x)$	$L = \int_a^b \sqrt{1+(f'(x))^2}\,dx$

$$f(x) = 3x^{2/3} - 10$$

$$f'(x) = 3\left(\tfrac{2}{3}\right)x^{-1/3} = 2x^{-1/3}$$

$$\boxed{L = \int_8^{27} \sqrt{1+(2x^{-1/3})^2}\,dx}$$

The challenge with arc length problems is often the level of integration skills required to finish the problem.

$$L = \int_8^{27} \sqrt{1+\frac{4}{x^{2/3}}}\,dx = \int_8^{27} \sqrt{\frac{x^{2/3}+4}{x^{2/3}}}\,dx$$

$$= \int_8^{27} x^{-1/3}\sqrt{x^{2/3}+4}\,dx$$

This can be solved using u-substitution

$$(x^{2/3}+4)^{1/2}$$

$u = x^{2/3}+4$

$du = \frac{2}{3}x^{-1/3}dx$

$\frac{3}{2}du = x^{-1/3}dx$

$$\int (u)^{1/2}\left(\tfrac{3}{2}du\right) = \frac{3}{2}\int u^{1/2}du = \frac{3}{2}\cdot\frac{2}{3}u^{3/2}$$

$$= (x^{2/3}+4)^{3/2}\Big|_8^{27} = (27^{2/3}+4)^{3/2} - (8^{2/3}+4)^{3/2}$$

$$= \boxed{13^{3/2} - 8^{3/2}} \approx \boxed{24.2}$$

8.1
③

Example 2

Set up an integral for finding the arc length of the graph of $y = y^3 - x$ from $A(0,-1)$ to $B(6,2)$.

$y = y^3 - x$

$y - y^3 = -x$

$-x = -y^3 + y$

$x = +y^3 - y$

form

$x = g(y)$

form of choice

$$L = \int_a^b \sqrt{1 + (g'(y))^2}\, dy$$

$x = +y^3 - y$

$\frac{dx}{dy} = g'(y) = 3y^2 - 1$

$(3y^2-1)^2 = 9y^4 - 6y^2 + 1$

$$L = \int_{-1}^{2} \sqrt{1 + (-3y^2-1)^2}\, dy$$

$$L = \int_{-1}^{2} \sqrt{9y^4 - 6y^2 + 2}\, dy$$

Example 3

Find the arc length of $30xy^3 - y^8 = 15$ from $\left(\frac{8}{15}, 1\right)$ to $\left(\frac{271}{240}, 2\right)$

$30xy^3 = 15 + y^8$

option 1
solve for x

option 2
solve for y

prep work

*

$30xy^3 = 15 + y^8$

$x = (15+y^2)\left(\frac{1}{30y^3}\right) = \frac{15+y^8}{30y^3}$

Quotiet Rule

$f = 15 + y^8 \quad g = 30y^3$

$f' = 8y^7 \quad g' = 90y^2$

$$\frac{dx}{dy} = \frac{8y^7(30y^3) - 90y^2(15+y^8)}{(30y^3)^2} = \frac{240y^{10} - 1350y^2 - 90y^{10}}{900y^6}$$

$$\frac{dx}{dy} = \frac{150y^{10} - 1350y^2}{900y^6} = \frac{15y^8 - 135}{90y^4} = \frac{3y^8 - 27}{18y^4} = \frac{y^8 - 9}{6y^4}$$

$$\left(\frac{y^8-9}{6y^4}\right)^2 = \frac{y^{16} - 18y^8 + 81}{36y^8}$$

$$L = \int \sqrt{1 + (g'(y))^2}\, dy$$

$$L = \int_1^2 \sqrt{1 + \frac{y^{16} - 18y^8 + 81}{36y^8}}\, dy$$

$$L = \int_1^2 \sqrt{\frac{36y^8 + y^{16} - 18y^8 + 81}{36y^8}}\, dy = \int_1^2 \sqrt{\frac{18y^8 + y^{16} + 81}{36y^8}}\, dy$$

8.1
5

$$L^2 = \int_1^2 \left(\frac{y^{16} + 18y^8 + 81}{36y^8} \right) dy$$

$$L^2 = \int_1^2 \left(\frac{(y^8+9)^2}{36y^8} \right) dy$$

trick/tip $\quad L = \int_1^2 \sqrt{\frac{(y^8+9)^2}{(6y^4)^2}}\, dy$

$$L = \int_1^2 \frac{y^8+9}{6y^4} dy$$

$$L = \int_1^2 \left(\frac{y^8}{6y^4} + \frac{9}{6y^4} \right) dy = \int_1^2 \left(\frac{1}{6}y^4 + \frac{3}{2}y^{-4} \right) dy$$

$$L = \frac{1}{6}\frac{y^5}{5} + \frac{3}{2}\frac{y^{-3}}{-3} \Bigg|_1^2 = \frac{1}{30}(y^5) - \frac{1}{2}\left(\frac{1}{y^3}\right) \Bigg|_1^2$$

$$L = \left(\frac{1}{30}(2)^5 - \frac{1}{2}\left(\frac{1}{(2)^3}\right) \right) - \left(\frac{1}{30}(1)^5 - \frac{1}{2}\left(\frac{1}{(1)^3}\right) \right)$$

$$L = \underset{2\cdot3\cdot5}{\frac{32}{30}} - \underset{2^4}{\frac{1}{16}} - \underset{2\cdot3\cdot5}{\frac{1}{30}} + \underset{2}{\frac{1}{2}}$$

lcd. $2^4 \cdot 3 \cdot 5 = 240$

$$L = \frac{32(8) - 1(15) - 1(8) + 1(120)}{240} = \boxed{\frac{353}{240} \approx 1.47}$$

Arc Length

Section 8.1
Practice Problems

Find the arc length for each function

1. $8x^2 = 27y^3$ from $(1, \frac{2}{3})$ to $(8, \frac{8}{3})$
2. $y = 5 - \sqrt{x^3}$ from $(1, 4)$ to $(4, -3)$
3. $x = \frac{y^4}{16} + \frac{1}{2y^2}$ from $(\frac{9}{8}, -2)$ to $(\frac{9}{16}, -1)$

Find an equation to determine the arc length for each given function. DO NOT SOLVE.

4. $2y^3 - 7y + 2x - 8 = 0$ from $(3, 2)$ to $(4, 0)$
5. $11x - 4x^3 - 7y + 7 = 0$ from $(1, 2)$ to $(0, 1)$

Arc length
Section 8.1
Practice Problems

answers and solution hints

1. one option

$y = \left(\frac{2}{3}\right)x^{2/3}$

$$L = \int_1^8 \sqrt{1 + \left(\frac{16}{81}\right)x^{-2/3}}\,dx = \int_1^8 x^{-1/3}\sqrt{x^{2/3} + \frac{16}{81}}\,dx$$

u-substitution

final answer: $\left(4 + \frac{16}{81}\right)^{3/2} - \left(1 + \frac{16}{81}\right)^{3/2}$

2. $\frac{8}{27}\left(10^{3/2} - \frac{13^{3/2}}{4^{3/2}}\right)$

3. final answer $\frac{21}{16}$ note that $\sqrt{a^2} = |a| = -a$ if $a < 0$

follow example 3

$$-\left[\frac{y^4}{16} - \frac{1}{2y^2}\right]_{-2}^{-1} = \frac{21}{16}$$

4. $\int_0^2 \left(\frac{53}{4} - 21y^2 + 9y^4\right)^{1/2} dy$ find lcd

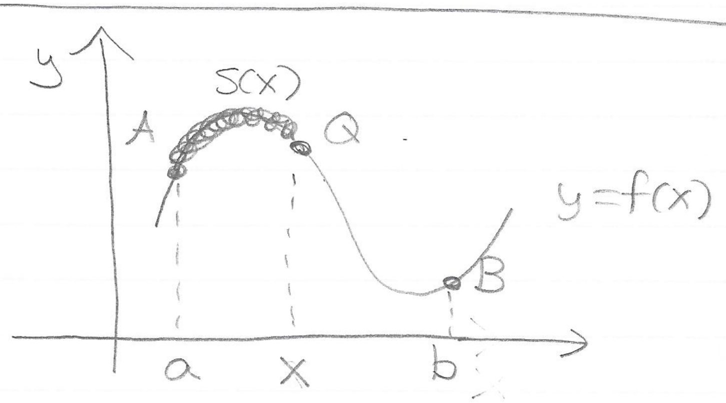
5. $\frac{1}{7}\int_0^1 \sqrt{144x^4 - 264x^2 + 170}\,dx$ find lcd

Extra material | something to think about

Let's look at arc length from another point of view. This will help us to understand surface area in section 8.2

Let the function f be smooth on $[a, b]$. The arc length function (called s) for the graph of f on $[a, b]$ is given by

$$s(x) = \int_a^x \sqrt{1 + [f'(t)]^2}\, dt \qquad \text{where } a \le x \le b$$



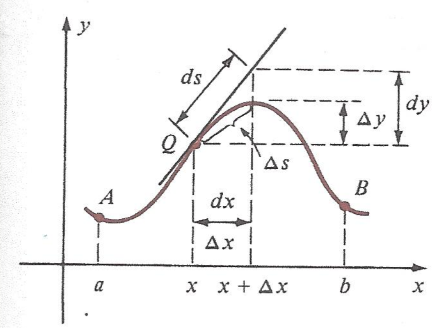


FIGURE 6.42 $(ds)^2 = (dx)^2 + (dy)^2$

Let f be a smooth curve on $[a, b]$ and let s be the arc length function for the graph of $y = f(x)$ on $[a, b]$. If dx and dy are differentials of x & y, then

1) $ds = \sqrt{1 + [f'(x)]^2}\ dx$

2) $(ds)^2 = (dx)^2 + (dy)^2$

ds = approximation for arc length.

Example | using differentials to find arc length

Let $y = x^3 + 2x$ from $(1,3)$ to $(1.2, 4.128)$

$y = x^3 + 2x$

from previous theorem

$f(x) = x^3 + 2x$

$x = 1$

$ds = \sqrt{1 + (f'(x))^2}\, dx$

$dx = 1.2 - 1 = 0.2$

$ds = \sqrt{1 + (3x^2 + 2)^2}\, dx$

ds = approximation for the arch length

$ds = \sqrt{1 + (3x^2+2)^2}\,(0.2)$ with $x = 1$

$ds = \sqrt{1 + (3(1)^2+2)^2}\,(0.2)$

$= \sqrt{26}\,(0.2)$

$\approx \boxed{1.02}$